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The $SU(2)$ sector in AdS/CFT

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Abstract

In the large N limit of $\mathcal{N} = 4$ Super Yang-Mills, the mixing under dilatations of the $SU(2)$ sector, single trace operators composed of L complex scalar fields of two types, is closed to all orders in perturbation theory. By relying on the AdS/CFT correspondence, and by examining the currents for semiclassical strings, we present evidence which implies that there are small mixings that contradict the closure of the $SU(2)$ sector in the strong coupling limit. These mixings first appear to second order in the λ/L^2 expansion.

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Large N gauge theories are dominated by planar diagrams, the diagrams looking like world-sheets swept out by a string [1]. The string theory "dual" to a particular 4 dimensional gauge theory lives on a curved, higher dimensional manifold [2]. The formulation of this duality was made precise in the case of $\mathcal{N} = 4$ Super Yang-Mills (SYM), where it was argued that the string dual is type IIB propagating on an $AdS_5 \times S_5$ target space with Ramond-Ramond 5-form background flux [3, 4, 5]. Among its many consequences, the AdS/CFT correspondence equates the dimensions of the gauge invariant operators with the energies, conjugate to the global time coordinate, of the dual string states in appropriate units.

In the large N limit, we can restrict the gauge invariant operators to be linear combinations of single trace operators. The problem of solving for the dimensions of these operators can then be mapped to the problem of solving for the energies of a one-dimensional spin chain, the different fields in the $PSU(2,2|4)$ singleton multiplet corresponding to the possible "spins" at each site [6, 7, 8]. The hamiltonian for the spin chain is the dilatation operator of the full superconformal algebra.

In order to simplify the problem, it is convenient to look for closed sectors of $PSU(2,2|4)$. The simplest nontrivial sector is the $SU(2)$ sector, where the operators are restricted to contain only two types of complex scalar fields, say Z and W. For a massless theory, dilatations only mix operators with the same bare dimension. So if one considers an operator $\mathcal{O}$ that is a single trace composed of J_1 Z fields and J_2 W fields, then it is straightforward to show using the R-symmetry and Lorentz invariance that dilatations can only mix $\mathcal{O}$ with operators which have the same number of Z and W fields *to all orders in perturbation theory*. In the $SU(2)$ sector, the one loop dilatation operator is equivalent to the Heisenberg spin chain with nearest neighbor interactions. At n loop order, the interactions between the spins range over n sites, and the dilatation operator restricted to this sector is known, with a few assumptions, to five loop order and is consistent with integrability [9, 10, 11],

In order to test the AdS/CFT duality, one needs to find the eigenvalues of the dilatation operator acting on operators with $L = J_1 + J_2$ total fields, and then compare them to the energies of the corresponding string states. This program has met with apparent success in the BMN limit [12, 13, 14], where the operators are very close to being BPS. It has also been successful, to a point, in the long wave-length limit, where the eigenvalues of the dilatation operator have been shown to match the string predictions at the one-loop level [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40], and in some cases up to two loop order [42, 24, 43, 44, 45],

although there is a mismatch at three loop order [46, 42, 47, 44]. Both the BMN limit and the long wave-length limit can be treated using a semiclassical analysis [48, 49], with expansion parameter λ/L^2, where λ is the 't Hooft parameter and quantum corrections are suppressed by $1/L$.

In any case, we can now ask the following question: is there a closed $SU(2)$ sector for the dual string?[1] The closure of the $SU(2)$ sector in $\mathcal{N}=4$ SYM is shown at the perturbative level. However, computations for the dual string are best under control when the coupling is large, where perturbative statements can break down.

In SYM, the dilatation operator becomes more and more complicated as we go to higher orders in perturbation theory, and it appears unlikely that an all orders expression will be found [2]. Nevertheless, we do know that there are approximately $2^L/L$ independent single trace $SU(2)$ sector operators in the large N limit[3]. If one could establish a closed $SU(2)$ sector on the string side, then one would also like to see the $2^L/L$ string states that are dual to these operators. Clearly most of these states are outside the long wave-length limit, so a full quantization of the string theory is required.

Let us now examine the question of $SU(2)$ closure in the dual string picture, considering semiclassical states that are dual to long wavelength operators. Since the operators dual to the strings have only two types of scalar fields and no covariant derivatives, the semiclassical string motion is constrained to an $R\times S_3$ subspace of the full $AdS_5\times S_5$. The isometry group of S_3 is $SO(4)=SU(2)_L\times SU(2)_R$ where we have labeled the $SU(2)$ subgroups in terms of left and right currents on the $SU(2)$ sigma model. We can express the coordinates on S_3 in terms of an $SU(2)$ group element

$$g=\begin{pmatrix} Z & W \\ -\overline{W} & \overline{Z}\end{pmatrix}, \tag{1}$$

where

$$|Z|^2+|W|^2=1. \tag{2}$$

The sigma model action is invariant under two global $SU(2)$ transformations, $g(\tau,\sigma)\rightarrow$

[1] My talk at the RTN conference was a review of the use of spin chains to compute anomalous dimensions in SYM, with the goal of comparing these results to string theory. Since there are already excellent published reviews on this subject [50, 11], and since many of the questions were about the three loop disagreement between gauge theory and string theory predictions, I have decided to present in these precedings some previously unpublished work about another comparision between SYM and semiclassical string results.

[2] Although it might be possible to find the S-matrix for the spin chain to all orders [45].

[3] For the spin chain there are 2^L states, but for the gauge theory there is a trace condition that reduces this number somewhat.

$g(\tau,\sigma)U$ and $g(\tau,\sigma)\to Vg(\tau,\sigma)$. Hence, we can construct a right current, $j_\alpha=-ig^{-1}\partial_\alpha g$, and a left current, $\ell_\alpha=-i\partial_\alpha g g^{-1}$.

If we treat the sigma model classically, then it is convenient to treat $j_1^a(\sigma)$ and $\ell_1^a(\sigma)$ as canonical coordinates [51], whose Poisson brackets then satisfy[4]

$$\{j_1^a(\sigma),j_1^b(\sigma')\}=\{\ell_1^a(\sigma),\ell_1^b(\sigma')\}=\{j_1^a(\sigma),\ell_1^b(\sigma')\}=0. \tag{3}$$

The time-like components of the currents are not precisely the canonical momenta, but instead have Poisson brackets given by

$$\begin{aligned}
\{j_0^a(\sigma),j_1^b(\sigma')\} &= i\varepsilon^{abc}j_1^c(\sigma)\delta(\sigma-\sigma')+\delta^{ab}\delta'(\sigma-\sigma')\\
\{\ell_0^a(\sigma),\ell_1^b(\sigma')\} &= i\varepsilon^{abc}\ell_1^c(\sigma)\delta(\sigma-\sigma')+\delta^{ab}\delta'(\sigma-\sigma')\\
\{j_0^a(\sigma),\ell_1^b(\sigma')\} &= ig^{-1}(\sigma)t^a g(\sigma')\delta'(\sigma,\sigma'),
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\{j_0^a(\sigma),j_0^b(\sigma')\} &= i\varepsilon^{abc}j_0^c(\sigma)\delta(\sigma-\sigma')\\
\{\ell_0^a(\sigma),\ell_0^b(\sigma')\} &= i\varepsilon^{abc}\ell_0^c(\sigma)\delta(\sigma-\sigma')\\
\{j_0^a(\sigma),\ell_0^b(\sigma')\} &= 0\,,
\end{aligned} \tag{5}$$

where t^a are $SU(2)$ generators satisfying

$$[t^a,t^b]=i\varepsilon^{abc}t^c,\qquad\qquad \mathrm{Tr}(t^at^b)=\frac{1}{2}\delta^{ab}. \tag{6}$$

Notice that the left and right currents are not completely independent.

From the Poisson brackets in (5), we can start to see why the spin chain is related to the sigma model [20]. Suppose that we have a one dimensional chain with L sites and two independent spin operators at each site k, $\vec{S}_k$ and $\vec{T}_k$, satisfying the commutation relations

$$[S_k^a,S_{k'}^b]=i\varepsilon^{abc}S_k^c\delta_{kk'}\,,\qquad [T_k^a,T_{k'}^b]=i\varepsilon^{abc}T_k^c\delta_{kk'}\,,\qquad [S_k^a,T_{k'}^b]=0. \tag{7}$$

In the limit that $L\to\infty$, the commutation relations in (7) would match the quantized version of (5), where the Poisson brackets are replaced by Dirac brackets. Or put another way, we can identify the expectation values of S_k^a and T_k^a with the classical currents as

$$\langle S_k^a\rangle = j_0^a\,(2\pi k/L)\,,\qquad\qquad \langle T_k^a\rangle=\ell_0^a\,(2\pi k/L)\,. \tag{8}$$

[4]In [52] an alternative set of Poisson relations has been proposed, which necessarily uses a different Hamiltonian for the time evolution of the currents. This Hamiltonian naturally appears as one of the local charges in the sigma model [53]. I thank K. Zarembo for remarks on this point.

Returning to the components of g, (Z, W) and $(-\overline{W}, \overline{Z})$ transform as doublets under $SU(2)_R$ transformations, while $(Z, -\overline{W})$ and $(W, \overline{Z})$ transform as doublets under $SU(2)_L$ transformations. Hence, for operators in the $SU(2)$ sector, the $SU(2)_L$ spin is up at every site. This suggests that the desired classical solutions should have a left current satisfying

$$\ell_0 = \begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}. \tag{9}$$

where ℓ is assumed to be a function of τ and σ [5]. This would correspond to all left charges aligning in the same direction, indicating that the dual operator only has Z and W fields and not their complex conjugates.

But the condition in (9) is overly restrictive. The condition that ℓ_0 have no off-diagonal terms immediately leads to the relation $Z = f(\sigma)W$. This, along with (2), means that

$$\partial_0(|Z|^2) = \partial_0(|W|^2) = 0. \tag{10}$$

Hence, we can write $Z = g_1(\sigma)e^{iw(\tau)}$, $W = g_2(\sigma)e^{iw(\tau)}$, with

$$|g_1(\sigma)|^2 + |g_2(\sigma)|^2 = 1. \tag{11}$$

The relevant sigma model action in conformal gauge is

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \frac{1}{2} \left(\frac{1}{2}(\mathrm{Tr}(\ell_0\ell_0) - \mathrm{Tr}(\ell_1\ell_1)) - \partial_\alpha t \partial^\alpha t \right) \tag{12}$$

From the reparameterization invariance we can choose $t = \kappa\tau$, where κ is a constant, which leads to the constraint

$$\kappa^2 = \frac{1}{2}\Big(\mathrm{Tr}(\ell_0^2) + \mathrm{Tr}(\ell_1^2)\Big) = \left(\frac{dw}{d\tau}\right)^2 + |\bar{g}_1 g_1' + \bar{g}_2 g_2'|^2 + |g_1 g_2' - g_2 g_1'|^2. \tag{13}$$

Since g_1, g_2 and κ are independent of τ, we see that $w = w_0\tau$ and so ℓ_0 is also independent of τ. The other constraint is

$$\mathrm{Tr}(\ell_0\ell_1) = 0, \tag{14}$$

which gives

$$\bar{g}_1 g_1' + \bar{g}_2 g_2' = 0. \tag{15}$$

Finally the equation of motion is

$$\partial_0\ell_0 - \partial_1\ell_1 = 0. \tag{16}$$

[5] The τ and σ dependence is gauge dependent, but under an appropriate gauge choice, ℓ can be set to a constant L [43].

Since ℓ_0 is independent of τ, we have $\partial_1 \ell_1 = 0$, which leads to the relation

$$g_1 g_2'' = g_2 g_1''. \tag{17}$$

Eqs. (15) and (17) can be satisfied if either g_1 or g_2 is zero. Such a solution would correspond to a point like string which is dual to the chiral primary operator $\mathrm{Tr}(Z^L)$. If neither g_1 nor g_2 is zero, then it is convenient to write these functions as $g_j = \rho_j e^{i\theta_j}$ where ρ_j and θ_j are real functions of σ, and so

$$\rho_1^2 + \rho_2^2 = 1. \tag{18}$$

Then from (15) we find that

$$\rho_1^2 \theta_1' = -\rho_2^2 \theta_2', \tag{19}$$

and from (17) we also find

$$(\rho_1^2 \theta_1')' = (\rho_2^2 \theta_2')' = 0. \tag{20}$$

Plugging (19) and (20) into (13) gives the general solution

$$\rho_1^2 = \frac{1}{2}(1 + A\cos(2m\sigma + \delta)), \tag{21}$$

with

$$\kappa^2 = w_0^2 + m^2\,, \tag{22}$$

where m is an integer and $-1 < A < 1$. Hence, these solutions correspond to the Frolov and Tseytlin circular string solutions with $J_1 = J_2$ [54]. In fact, the constant A can be set to zero by choosing a new basis for Z and W. Hence, except for the point-like and the Frolov-Tseytlin circular string, it is not possible to find a classical solution where the $SU(2)_L$ charge is aligned, that is proportional to σ_3, everywhere on the string.

It is instructive to write the left currents in terms of angles on S_3. If we express Z and W as

$$Z = \cos\theta e^{i\phi_1} \qquad W = \sin\theta e^{i\phi_2}\,, \tag{23}$$

then the left current can be written as

$$\ell_0 = \begin{pmatrix} \dot\varphi_1 + \dot\varphi_2 \cos 2\theta & (-i\dot\theta + \dot\varphi_2 \sin 2\theta) e^{2i\varphi_1} \\ (i\dot\theta + \dot\varphi_2 \sin 2\theta) e^{-2i\varphi_1} & -\dot\varphi_1 - \dot\varphi_2 \cos 2\theta \end{pmatrix}, \tag{24}$$

where $\varphi_{1,2} = \frac{1}{2}(\phi_1 \pm \phi_2)$. The condition that ℓ_0 have no off-diagonal terms leads to $\dot\theta = \dot\varphi_2 = 0$, which are precisely the conditions found by Kruczenski for the restriction of

the σ-model to match the spin chain effective action at one-loop [20]. However, because of the constraints, these restrictions on $\dot{\theta}$ and $\dot{\varphi}_2$ do not hold beyond the one-loop level.

To see what happens beyond the one-loop level, let us consider a particular class of solutions where the charge is uniformly distributed on the string in conformal gauge [43]. For these solutions we choose the ansatz [55, 56]

$$Z = \cos\theta_0 e^{iw_1\tau + m_1\sigma} \qquad\qquad W = \sin\theta_0 e^{iw_2\tau + m_2\sigma}. \tag{25}$$

The constraints lead to the equations

$$\begin{aligned} \kappa^2 &= \cos^2\theta(w_1^2 + m_1^2) + \sin^2\theta(w_2^2 + m_2)^2 \\ 0 &= w_1 m_1 \cos^2\theta + w_2 m_2 \sin^2\theta \end{aligned} \tag{26}$$

and the equation of motion leads to

$$w_1^2 - m_1^2 = w_2^2 - m_2^2. \tag{27}$$

Substituting into (24), we find

$$\ell_0 = \begin{pmatrix} \frac{1}{2}(w_1+w_2) + \frac{1}{2}(w_1-w_2)\cos 2\theta_0 & \frac{1}{2}(w_1-w_2)\sin 2\theta_0 e^{i(w_1+w_2)\tau + i(m_1+m_2)\sigma} \\ \frac{1}{2}(w_1-w_2)\sin 2\theta_0 e^{-i(w_1+w_2)\tau - i(m_1+m_2)\sigma} & -\frac{1}{2}(w_1+w_2) - \frac{1}{2}(w_1-w_2)\cos 2\theta_0 \end{pmatrix}, \tag{28}$$

From the equations of motion, we see that $w_1 - w_2 \approx (m_1^2 - m_2^2)/2w_1$ and $w_1 \approx L/\sqrt{\lambda}$. Hence, we see that $\ell_0 = \vec{\ell}_0 \cdot \vec{\sigma}$ where $|\ell_0^{1,2}| \sim \frac{\lambda}{L^2}|\ell_0^3|$. In other words, there is a small component of the left current that is not aligned in the $\hat{z}$ direction[6]. It is clear that this is a two loop effect, since the contribution to κ^2 from this term is $|\ell_0^1|^2 + |\ell_0^2|^2$. One can also show that the nondiagonal piece cannot be removed by choosing a different gauge for the Polyakov action, such as the nondiagonal gauge used in [43].

One way to interpret these results is that there is mixing outside of the $SU(2)$ sector that is nonperturbative in λ, with the string capturing this nonperturbative behavior. This could occur through a reversal in the order of limits, where on the string side one first takes $L \to \infty$ and then expands in λ/L^2, while in the gauge theory, one first expands

[6]We can also see that the the solution in (28) is of Landau-Lifschytz type. If we write $\ell_0 = \vec{T}\cdot\vec{\sigma}$, then we see that to two loop order, the components of $\vec{T}$ satisfy

$$\dot{\vec{T}} \approx \frac{1}{(m_1+m_2)^2}\vec{T}\times\vec{T}''. \tag{29}$$

It is not clear what the significance of this equation is, especially the meaning of the coefficient that depends on the winding.

in λ and then takes $L \to \infty$ [42, 24, 47], although I know of no specific process through which this could occur[7].

The nondiagonal component of the left current also leads to an uncertainty in what is meant by the bare dimension of the operator. In comparing string solutions to Yang-Mills solutions, one usually assumes that the bare dimension is the left charge, since on the Yang-Mills side there are only Z and W scalar fields. However, if there are also $\bar{Z}$ and $\bar{W}$ fields, then the bare dimension would be larger than L, since L is measuring the net number of unbarred fields minus the number of barred fields. If the bare dimension is larger, then this means that the anomalous dimension is smaller at the two loop level and so there would be a mismatch between the gauge theory and string theory predictions.

It is also possible that the $SU(2)$ sector *is* closed at strong coupling, but there is a subtlety in identifying the left and right currents with the spins on the chain as in (8). However, the identification in (8) seems quite natural and there is no obvious alternative.

We can also examine the possible mixing by looking at the finite gap solutions of the principle chiral model. In [24] a one to one map was found between the long wave-length solutions of the $SU(2)$ sector and the corresponding solutions for the sigma model. This work was later extended into other sectors [34, 35, 36, 37]. The relevant facts can be found in [24], but on the SYM side, the one-loop Bethe equations can be reduced to solving for a resolvant,

$$G(\varphi) = \int \frac{d\varphi' \rho(\varphi')}{\varphi - \varphi'}\,, \tag{30}$$

where $\rho(\varphi)$ is the density of Bethe roots in the complex plane of the spectral parameter φ. $G(\varphi)$ has square root branch cuts and asymptotic behavior

$$G(\varphi) \sim \frac{J_2/L}{\varphi} \qquad\qquad \varphi \to \infty. \tag{31}$$

$G(\varphi)$ is also the generator of the conserved charges [18, 19], with $G(0) = -2\pi m$ being the total momentum of the impurities in the spin chain, and $G'(0) = 8\pi^2 L\gamma/\lambda$ giving the anomalous dimension.

In the long wave-length limit, it is also possible to consider larger classes of scalar operators that live in an $SO(4)$ sector. Strictly speaking, this sector does not close, even at one-loop, but in the long-wave length limit, this nonclosure is $1/L$ effect and can be ignored [44]. These $SO(4)$ solutions can then be considered as two independent $SU(2)$ chains, with the resolvant being the sum of the two, $G(\varphi) = G_1(\varphi) + G_2(\varphi)$ where $G_{1,2}(\varphi)$

[7] I thank M. Staudacher for remarks on this point.

is the resolvant for each $SU(2)$. Since each $G_i(\varphi)$ is a generator of conserved charges, we could also consider $\tilde{G}(\varphi) = G_1(\varphi) - G_2(\varphi)$. However, the anomalous dimension is in $G(\varphi)$ so this is the one we are interested in.

Higher orders in λ will modify the equations that the $G_i(x)$ satisfy, but in the long wave-length limit, we still expect the $SU(2)$ sectors to remain separated. Hence, the various solutions can be written as functions on a 4 sheeted surface, with branch cuts connecting the first surface to the second and the third surface to the fourth.

On the string side, there is an analog of $G(x)$. The σ-model on $R \times S_3$ is classically integrable [57][8]. This means that there is a lax pair

$$\begin{aligned} \mathcal{L} &= \partial_\sigma - \frac{i\sqrt{T}}{2}\left(\frac{j_+}{x-\sqrt{T}} + \frac{j_-}{x+\sqrt{T}}\right) \\ \mathcal{M} &= \partial_\tau - \frac{i\sqrt{T}}{2}\left(\frac{j_+}{x-\sqrt{T}} - \frac{j_-}{x+\sqrt{T}}\right) \end{aligned}$$

that satisfies the flatness condition $[\mathcal{L}, \mathcal{M}] = 0$, which is a consequence of the equation of motion

$$\partial_+ j_- + \partial_- j_+ = 0 \tag{32}$$

and the identity

$$\partial_+ j_- - \partial_- j_+ + i[j_+, j_-] = 0. \tag{33}$$

The parameter T is $\frac{\lambda}{16\pi^2 L^2}$ and x is a spectral parameter. We can then look for the solution of the equation

$$\mathcal{L}\psi = 0, \tag{34}$$

which is formally given by the path ordered expression

$$\psi(\sigma, x, \tau) = \mathcal{P}\exp\left(\frac{i\sqrt{T}}{2}\int_0^\sigma d\sigma'\left(\frac{j_+}{x-\sqrt{T}} + \frac{j_-}{x+\sqrt{T}}\right)\right). \tag{35}$$

Setting $\Omega(x) = \psi(2\pi, x, \tau)$, it is immediately clear that Ω is constant in τ because of the flatness condition. Under a similarity transformation, $\Omega(x)$ can be diagonalized to

$$\Omega(x) = \begin{pmatrix} e^{ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix}, \tag{36}$$

where $p(x)$ can have square root branch cuts, the branch points occurring at values of x where $e^{ip(x)} = \pm 1$.

[8]The σ-model on the full $AdS_5 \times S_5$ is also classically integrable [58, 59, 60, 55, 56, 61, 62, 63, 41].

Asymptotically, $p(x)$ behaves as

$$p(x) \sim -\frac{\pi\kappa\sqrt{T}}{x \mp \sqrt{T}} \qquad x \to \pm\sqrt{T} \tag{37}$$

$$p(x) \sim -\frac{Q_R}{2Lx} \qquad x \to \infty \tag{38}$$

and

$$p(x) = 2\pi m + \frac{Q_L}{2LT}x + \ldots \qquad x \to 0, \tag{39}$$

where $Q_{R,L}$ are the left and right charges

$$\begin{aligned} Q_R &= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} j_0^3 d\sigma \\ Q_L &= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \ell_0^3 d\sigma \,. \end{aligned} \tag{40}$$

We can then define $G_s(x) = p(x) + \frac{(\Delta/L)x}{2(x^2-T)}$ which has no poles on one of the sheets. $G_s(x)$ has square root branch cuts on a two-sheeted surface, but it has information about both $SU(2)_R$ and $SU(2)_L$. If all branch points are at $|x| >> \sqrt{T}$ as $T \to 0$, then it was shown in [24] that $G_s(x)$ approaches $G_1(x)$. But it was also shown in [35] that there is an inversion symmetry, such that $G(x) \to -G(T/x) + 2\pi n$ switches $SU(2)_R$ with $SU(2)_L$. Hence, if all branch points are located inside $|x| << \sqrt{T}$, then $-G_s(T/x) + 2\pi n$ approaches $G_2(x)$. However, if there are branch points at both extremes, then we should consider the total contribution, and so we would have

$$G(x) = G_1(x) + G_2(x) = G_s(x) - G_s(T/x) + 2\pi n \qquad T \to 0. \tag{41}$$

Two simple examples were given in [24]. The first corresponds to the Frolov-Tseytlin solution with $J_1 = J_2$. In this case

$$G_s(x) = \frac{1}{2}\frac{x}{x^2 - T}\left(\Delta/L + \sqrt{16\pi^2 m^2 x^2 + 1}\right) - 2\pi m, \tag{42}$$

where m is the winding and $\Delta = \sqrt{L^2 + m^2\lambda}$. Hence,

$$G_1(x) = \frac{1}{2x}\left(1 + \sqrt{16\pi^2 m^2 x^2 + 1}\right) - 2\pi m\,, \tag{43}$$

and $G_2(x) = 0$ as $T \to 0$. Note that the separation into the $SU(2)_R$ or $SU(2)_L$ sectors breaks down when $m^2 \sim L^2/\lambda$, which is also where the semiclassical string approximation breaks down. On the SYM side, this corresponds to the case where the anomalous dimension is of order L, the same order as the bare dimension.

The second example is for a pulsating string, where $G_s(x)$ is

$$G_s(x) = \frac{1}{2}\frac{x}{x^2-T}\left(\Delta/L + \sqrt{\left(2\pi m(x-T/x) - \frac{J}{L}\right)^2 + \frac{\Delta^2 - J^2}{L^2}}\right) - \pi m, \qquad (44)$$

where $J = Q_R = Q_L$. Hence, we find that

$$\begin{aligned} G_1(x) &= \frac{1}{2x}\left(1 + \sqrt{(2\pi m x - J/L)^2 + 1 - J^2/L^2}\right) - \pi m \\ G_2(x) &= \frac{1}{2x}\left(1 + \sqrt{(2\pi m x + J/L)^2 + 1 - J^2/L^2}\right) + \pi m \end{aligned} \qquad (45)$$

as $T \to 0$.

It is hence clear that the left and right pieces have effectively separated in the limit $T \to 0$. Essentially we have taken our two sheeted surface and cut a small hole of radius $\sqrt{T}$ out of each sheet. In the limit that $T \to 0$, the disc and the sheet with the hole in it separate into two sheets with no cuts connecting them[9]. Hence, our two-sheeted surface has separated into four sheets, with cuts only connecting the separate pairs. This can also be seen in the context of the full $SO(6)$ symmetry of the S_5 [35], where one starts with 4 sheets separated into pairs. The pairs are related by the inversion symmetry, the inversion taking the branch points inside $x = \sqrt{T}$ for one pair to the branch points outside $x = \sqrt{T}$ for the other pair.

Once we start considering higher order corrections in T, which correspond to higher orders in the Yang-Mills coupling, the separation can no longer be maintained. To see this, note that it was possible to successfully match all Frolov-Tseytlin solutions in string theory to SYM solutions up to two loop order in the $SU(2)$ sector [24], if one assumes that the left hand charge L is the bare dimension of the SYM operators. In order to demonstrate the matching, it was necessary to redefine the spectral parameter [24]. In [47, 64] it was shown how to do this in a very nice fashion. The idea is as follows: let us call the spectral parameter for the gauge theory φ and the spectral parameter for the string theory x. At the one-loop level, we have $\varphi = x$. Using (38), (39) and the definition of $G_s(x)$, we find that $G_s(x)$ has the asymptotic behavior

$$\begin{aligned} G_s(x) &\sim \frac{J}{Lx} + \frac{\Delta - L}{2Lx}, \qquad x \to \infty \\ G_s(x) &= 2\pi m - \frac{\Delta - L}{2LT}x + \ldots \qquad x \to 0, \end{aligned} \qquad (46)$$

[9] We are assuming that there are an even number of branchpoints both inside and outside the radius $\sqrt{T}$ so that there are no cuts running between the two regions. This is required in order to match to perturbative SYM. I thank N. Beisert for pointing out an incorrect statement on this point in the previous version.

where we have used that $Q_L = L$ and $Q_R = L - 2J$. We now assume that the string resolvant $G_s(x)$ can be written as an integral over a density of roots, $\rho_s(x)$, as

$$G_s(x) = \int \frac{dx' \rho_s(x')}{x - x'} \tag{47}$$

and so we have

$$\begin{aligned} \int dx' \rho_s(x') &= \frac{J}{L} + \frac{\Delta - L}{2L} \\ T \int \frac{dx' \rho_s(x')}{x'^2} &= \frac{\Delta - L}{2L}. \end{aligned} \tag{48}$$

For the gauge theory, we expect

$$\int d\varphi' \rho(\varphi') = \frac{J}{L}, \tag{49}$$

and so we can make the identification $\varphi = x + T/x$ and $\rho_s(x) = \rho_1(\varphi)$ [47]. Now let us see what this means for the relations between the resolvants. We have that [35]

$$\begin{aligned} G_1(\varphi) &= \int \frac{d\varphi' \rho_1(\varphi')}{\varphi - \varphi'} = \int \frac{dx'(1 - T/x'^2)\rho_s(x')}{(x - x')(1 - T/xx')} \\ &= \int dx' \rho_s(x') \left(\frac{1}{x - x'} + \frac{1}{T/x - x'} + \frac{1}{x'} \right) \\ &= G_s(x) + G_s(T/x) - G_s(0), \end{aligned} \tag{50}$$

where $x = \frac{1}{2}(\varphi + \sqrt{\varphi^2 - 4T})$ [10]. The result in (50) was shown to match the Yang-Mills result at the two loop level, but fails at three loops.

But we can also see a problem with (50) at the two loop level. If there are branch points in the region $|x| << \sqrt{T}$, then we would want to use $G(\varphi) = G_s(x) - G_s(T/x) + 2\pi n$ in order to match the *one-loop* result for the full $SU(2)_L \times SU(2)_R$ theory. Presumably, we can smoothly adjust the solution such that the branch points in $|x| << \sqrt{T}$ disappear, reducing everything to a single $SU(2)$. But then we see that we have the opposite sign for $G_s(T/x)$ as compared to (50). If we went further with this definition of $G(\varphi)$, then we would find that in the limit of small T, $G(\varphi)$ would have a pole at $\varphi = 0$ whose residue is linear in T, and hence $G(\varphi)$ would not match the gauge theory result at this order. So from this point of view, the two-loop matching condition seems unnatural.

[10] Note that (50) gives the same result if we choose the other solution, $x = \frac{1}{2}(\varphi - \sqrt{\varphi^2 - 4T})$, since this is T/x of the original solution.

Nevertheless, a prescription has been given to match an $SU(2)_L \times SU(2)_R$ solution at two loops [44], at least such that the dimension of the operator matches the energy of the string. We do not yet know how to show matching of *all* conserved quantities, although perhaps this can be accomplished with a different relation between φ and x.

In conclusion, the viewpoint that one should probably take, assuming that the semi-classical string results are trustworthy, is that it is not sensible to split up the dimension of an operator into a "bare" and an anomalous piece at strong coupling. If it were possible, then one could infer that there are a fixed number of fields, which would effectively keep the $SU(2)$ sector closed. Instead, only the full dimension is well defined and it is a function of $\sqrt{\lambda}$ and the charge L. In going from weak to strong coupling, the long wave-length operators in the $SU(2)$ sector smoothly flow to the Frolov-Tseytlin solutions, such that the first two orders in λ/L^2 agree.

As one gets away from the long wavelength limit, it becomes harder to identify the string states which will smoothly flow to operators in the $SU(2)$ sector as λ goes from strong to weak coupling. It may be that counting the $2^L/L$ string states is possible only if λ is small, where of course the string is strongly coupled. A proposal has been made to discretize the string Bethe equations, and use this as a quantized Bethe ansatz for the string states that flow from the $SU(2)$ sector [65]. Perhaps, one can count the solutions of these discretized Bethe equations by counting all Bethe strings of this model, as was done for the Heisenberg chain [66].

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